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Erratum

Erratum to: "On the set of discriminants of quadratic pairs" [JPAA 188 (2004) 33–44]

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In [1], the following result was announced (we refer to [1] for definitions and notation):

Theorem 1. *Let F be any field of characteristic* 2, *and let* (A, σ) *be a central simple F-algebra with a symplectic involution, which is not a division quaternion algebra. Then* $\partial(A, \sigma) = F/\wp(F)$.

Here, $\partial(A, \sigma)$ denotes the set of discriminants of quadratic pairs on A of the form (σ, f).

Unfortunately, the proof of the Key Lemma (and probably the Key Lemma itself), on which the proof of this theorem is based, is false. More precisely, the argument given to prove the isotropy of $q|_W$ is incorrect (for notation, see [1] or see the proof of Theorem 2).

However, a weaker version of this result is true. Precisely, we have:

Theorem 2. *Let F be any field of characteristic* 2, *and let A be a central simple F-algebra with a symplectic involution, which is not a division quaternion algebra. Then there exists a symplectic involution σ on A such that* $\partial(A, \sigma) = F/\wp(F)$. *In particular, any* $\alpha \in F/\wp(F)$ *is the discriminant of some quadratic pair on A.*

Let σ be a symplectic involution on A, and let $\ell_0 \in A$, such that $\ell_0 + \sigma(\ell_0) = 1$. Recall from [1] that in order to prove the equality $\partial(A, \sigma) = F/\wp(F)$, it is sufficient to prove that the quadratic form $q: s \in \mathrm{Sym}(A, \sigma) \mapsto \mathrm{Srd}_A(s) + \mathrm{Trd}_A(\ell_0 s)^2 \in F$ is universal. Recall that the radical of q is $\mathrm{Alt}(A, \sigma)$.

We first prove the following lemma.

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Lemma 3. *Let (A, σ) be a central simple F-algebra of degree $n=2m \geqslant 2$ with a symplectic involution. Assume that there exists $u, u' \in \mathrm{Sym}(A, \sigma)$ such that*

(1) $\mathrm{Span}_F(u, u') \cap \mathrm{Alt}(A, \sigma) = \{0\}$,
(2) $\mathrm{Span}_F(u, u')$ *is a totally isotropic two-dimensional subspace for the second trace form* $\mathscr{T}_{2,A}: a \in A \mapsto \mathrm{Srd}_A(a) \in F$.

Then we have $\partial(A, \sigma) = F/\wp(F)$.

Proof. Let $a_0 \in A$ such that $\mathrm{Trd}_A(a_0) = 1$. Then $\mathrm{Trd}_A(\ell_0(a_0 + \sigma(a_0))) = 1$. Let W be a subspace of $\mathrm{Sym}(A, \sigma)$ such that $\mathrm{Sym}(A, \sigma) = W \oplus \mathrm{Alt}(A, \sigma)$. Since the radical of q is $\mathrm{Alt}(A, \sigma)$, $q|_W$ is non-degenerate and the direct sum above is an orthogonal sum. Let $w_1, \ldots, w_n$ be a F-basis of W, and set $w_i' := w_i - \mathrm{Trd}_A(\ell_0 w_i)(a_0 + \sigma(a_0))$. Then the subspace W' generated by the w_i''s satisfies $\mathrm{Sym}(A, \sigma) = W' \perp \mathrm{Alt}(A, \sigma)$ and $\mathrm{Trd}_A(\ell_0 W') = \{0\}$. Hence we may assume that $\mathrm{Trd}_A(\ell_0 W) = \{0\}$, so $q|_W$ is just the restriction of $\mathscr{T}_{2,A}$ to W.

By assumption, the subspace $L := \mathrm{Span}_F(u, u') \oplus \mathrm{Alt}(A, \sigma)$ is a subspace of $\mathrm{Sym}(A, \sigma)$ of dimension $n(n-1)/2 + 2$. Since L contains $\mathrm{Alt}(A, \sigma)$, we have $L + W = \mathrm{Sym}(A, \sigma)$, hence, $\dim(L + W) = n(n+1)/2$. Since, $\dim W = n$, dimension count shows that $\dim(W \cap L) = 2$. Let w_1, w_2 be a basis of this intersection. Write $w_i = v_i + b_i + \sigma(b_i)$ for $v_i \in \mathrm{Span}_F(u, u')$ and $b_i \in A$. If $\mathrm{Trd}_A(b_i) = 0$ for some i, set $w = w_i$. If $\mathrm{Trd}_A(b_1)$ and $\mathrm{Trd}_A(b_2)$ are both non-zero, then set $w = \mathrm{Trd}_A(b_2)w_1 - \mathrm{Trd}_A(b_1)w_2$. In any case, w is a non-trivial element of $W \cap L$ such that $w = v + a + \sigma(a)$, with $v \in \mathrm{Span}_F(u, u')$ and $a \in A$ with $\mathrm{Trd}_A(a) = 0$.

We now proceed to show that $q|_W(w) = \mathrm{Srd}_A(w) = 0$. We have,

$$\begin{aligned}\mathrm{Srd}_A(w) = {} & \mathrm{Srd}_A(v) + \mathrm{Srd}(a + \sigma(a)) + \mathrm{Trd}_A(v)\mathrm{Trd}_A(a + \sigma(a)) \\ & + \mathrm{Trd}_A(v(a + \sigma(a))).\end{aligned}$$

Since $v \in \mathrm{Span}_F(u, u')$, we have $\mathrm{Srd}_A(v) = 0$ by assumption.

Moreover, $\mathrm{Trd}_A(a + \sigma(a)) = \mathrm{Trd}_A(a) + \mathrm{Trd}_A(\sigma(a)) = 2.\ \mathrm{Trd}_A(a) = 0$. We also have, $\mathrm{Trd}_A(v(a+\sigma(a))) = \mathrm{Trd}_A(va) + \mathrm{Trd}_A(v\sigma(a)) = \mathrm{Trd}_A(av) + \mathrm{Trd}_A(\sigma(v)\sigma(a)) = \mathrm{Trd}_A(av) + \mathrm{Trd}_A(\sigma(av)) = 0$. Finally, $\mathrm{Srd}(a + \sigma(a)) = \mathrm{Srd}_A(a) + \mathrm{Srd}_A(\sigma(a)) + \mathrm{Trd}_A(a)\mathrm{Trd}_A(\sigma(a)) + \mathrm{Trd}_A(a\sigma(a)) = 0$. Indeed we have $\mathrm{Srd}_A(a) = \mathrm{Srd}_A(\sigma(a))$, $\mathrm{Trd}_A(a) = 0$, and the reduced trace of any σ-symmetric element is 0, since σ is symplectic.

Then w is a non-zero isotropic vector for $q|_W$. Since $q|_W$ is non-degenerate, this implies that $q|_W \simeq q' \perp [0, 0]$, hence $q|_W$ is universal. The previous considerations then give the desired conclusion. $\square$

Now, the final argument in the "proof" of Theorem 1 in [1] shows the existence of a symplectic involution σ on A and of two σ-symmetric elements satisfying the conditions of the previous lemma, when A has index at least 4 (the remaining cases are considered in Section 3 of [1]). This concludes the proof of Theorem 2.

Remark 4. We are still convinced that Theorem 1 is true. Nevertheless, we do not know how to prove it for the moment.

References

[1] G. Berhuy, On the set of discriminants of quadratic pairs, J. Pure Appl. Algebra 188 (1–3) (2004) 33–44.